

# RECTANGULAR SEIFERT CIRCLES AND ARCS SYSTEM

TATSUO ANDO, CHUICHIRO HAYASHI AND MIWA HAYASHI

**ABSTRACT.** Rectangular diagrams of links are link diagrams in the plane  $\mathbb{R}^2$  such that they are composed of vertical line segments and horizontal line segments and vertical segments go over horizontal segments at all crossings. P. R. Cromwell and I. A. Dynnikov showed that rectangular diagrams of links are useful for deciding whether a given link is split or not, and whether a given knot is trivial or not. We show in this paper that an oriented link diagram  $D$  with  $c(D)$  crossings and  $s(D)$  Seifert circles can be deformed by an ambient isotopy of  $\mathbb{R}^2$  into a rectangular diagram with at most  $c(D) + 2s(D)$  vertical segments, and that, if  $D$  is connected, at most  $2c(D) + 2 - w(D)$  vertical segments, where  $w(D)$  is a certain non-negative integer.

In order to obtain these results, we show that the system of Seifert circles and arcs substituting for crossings can be deformed by an ambient isotopy of  $\mathbb{R}^2$  so that Seifert circles are rectangles composed of two vertical line segments and two horizontal line segments and arcs are vertical line segments, and that we can obtain a single circle from a connected link diagram by smoothing operations at the crossings regardless of orientation.

## 1. INTRODUCTION

Birman and Menasco introduced arc-presentation of links in [1], and Cromwell formulated it in [2]. Dynnikov pointed out in [3] and [4] that Cromwell's argument in [2] almost shows that any arc-presentation of a split link can be deformed into one which is “visibly split” by a finite sequence of elementary moves which do not change number of arcs of arc-presentations. He also showed that any arc-presentation of the trivial knot can be deformed into the trivial one with only two arcs by a finite sequence of merge elementary moves without increasing number of arcs. Since there are only finitely many arc-presentations with a fixed number of edges, these results give finite algorithms for the decision problems. As is shown in page 41 in [2], an arc-presentation is almost equivalent to a rectangular diagram.

A *rectangular diagram* of a link is a link diagram in the plane  $\mathbb{R}^2$  which is composed of vertical line segments and horizontal line segments such that no pair of vertical line segments are colinear, no pair of horizontal line segments are colinear, and the vertical line segment

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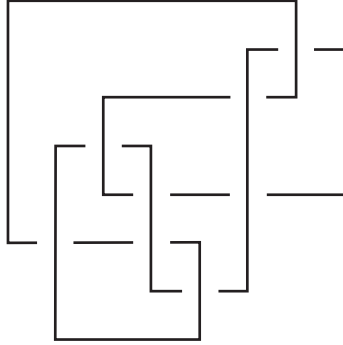


FIGURE 1. A rectangular diagram of the trivial knot with 8 vertical edges

passes over the horizontal line segment at each crossing. See Figure 1. These vertical line segments and horizontal line segments are called *edges* of the rectangular diagram. Every rectangular diagram has the same number of vertical edges and horizontal edges. It is known that every link has a rectangular diagram (Proposition in page 42 in [2]).

In [5], A. Henrich and L. Kauffman announced an upper bound of the number of Reidemeister moves needed for unknotting (Theorem 8) by applying Dynnikov's theorem to rectangular diagrams. Let  $D$  be an oriented link diagram in Morse form, and  $c(D)$ ,  $b(D)$  the numbers of crossings and maxima. Lemma 2 in [5] states that we can obtain a rectangular diagram with at most  $2b(D) + c(D)$  vertical edges from  $D$  by an ambient isotopy of the plane  $\mathbb{R}^2$ . In this paper, we consider the number of Seifert circles  $s(D)$  instead of that of maxima. Note that  $s(D)$  does not change under isotopy of  $\mathbb{R}^2$ . We obtain an estimation depending only on  $c(D)$ , too.

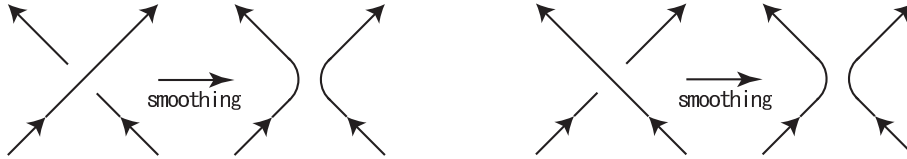


FIGURE 2. smoothing operation

Let  $D$  be an oriented link diagram in the plane  $\mathbb{R}^2$ . If we perform smoothing operations at all the crossings as shown in Figure 2, then we obtain a disjoint union of oriented circles in  $\mathbb{R}^2$  as in Figure 3, which we call *Seifert circles*. This operation is introduced by Seifert in [6] to construct an orientable surface spanning a knot.

A crossing  $x$  of a link diagram  $D$  in the plane  $\mathbb{R}^2$  is called *nugatory* if there is a circle  $C$  in  $\mathbb{R}^2$  which intersects  $D$  only in a single point at  $x$ . See Figure 4. Link diagrams with a nugatory crossing are often left out from consideration since we can get rid of a nugatory crossing by rotating the part of the link inside  $C$  through  $180^\circ$ .

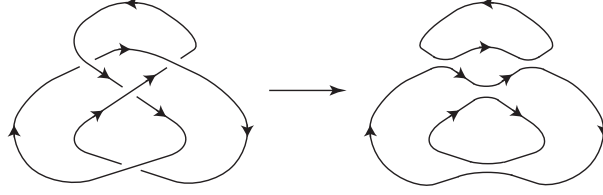


FIGURE 3. Seifert circles

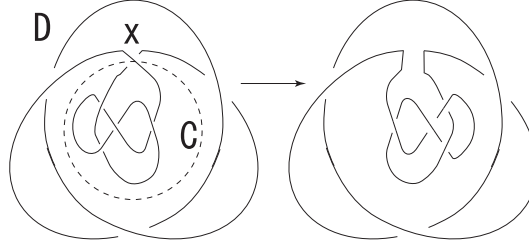


FIGURE 4. a nugatory crossing

**Theorem 1.1.** *Let  $D$  be an oriented link diagram in the plane  $\mathbb{R}^2$ , and  $c(D)$ ,  $s(D)$  the numbers of crossings and Seifert circles of  $D$  respectively. Then an adequate ambient isotopy of  $\mathbb{R}^2$  deforms  $D$  into a rectangular diagram with at most  $c(D) + 2s(D)$  vertical edges. If  $c(D) \geq 1$  and  $D$  has no nugatory crossings, then  $c(D) + 2s(D) - 2$  vertical edges are enough.*

A link diagram  $D$  in the plane  $\mathbb{R}^2$  is said to be *connected* if it is connected when the underpasses are restored at all the crossings.

An *undirected smoothing operation* at a crossing is a smoothing operations neglecting orientation of a ink. It may or may not respect the orientation of the link when we orient the link. By an adequate undirected smoothing operations, as shown in Figure 9, we obtain a single circle from any connected link diagram (Lemma 2.1). This leads to the next theorem.

**Theorem 1.2.** *Let  $D$  be a connected link diagram in  $\mathbb{R}^2$ , and  $c(D)$  the number of crossings of  $D$ . Then  $D$  can be deformed into a rectangular diagram with at most  $2c(D) - w(D) + 2$  vertical edges by an adequate ambient isotopy of  $\mathbb{R}^2$ , where  $w(d)$  is a non-negative integer defined as the width of  $D$  as below.*

In order to show Theorems 1.1 and 1.2, we observe a system of circles obtained by performing smoothing operations at all the crossings of a link diagram and arcs corresponding to crossings.

After we perform smoothing operations at all the crossings, we place a line segment connecting Seifert circles as a substitute for each crossing as shown in Figure 5 (1). Then we obtain a union of circles and arcs, which we call *Seifert circles and arcs system*. In Figure 5 (2), the one which is obtained from the knot diagram in Figure 3 is depicted.

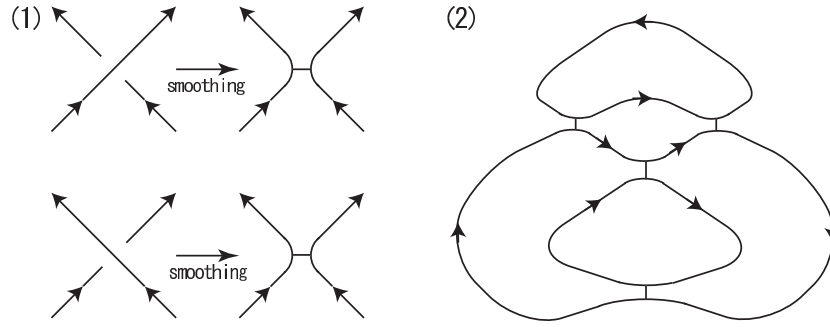


FIGURE 5. Seifert circles and arcs system

Note that Seifert circles and arcs system does not have an arc with its both endpoints in the same Seifert circle. (Otherwise, we would have a contradiction on orientation of the link.) Moreover, the orientations of the two circles containing the endpoints of an arc are both clockwise or both anti-clockwise if and only if one circle is contained in the disk bounded by the other.

When we apply undirected smoothing operations, we obtain a system of circles and arcs, where circles do not have orientations, and there may be an arc which has its both endpoints in the same circle.

In general, let  $C$  be a disjoint union of circles in the plane  $\mathbb{R}^2$ , and  $A$  a disjoint union of arcs in  $\mathbb{R}^2$  such that  $A \cap C = \partial A$ , where  $\partial A$  denotes the set of endpoints of arcs of  $A$ . Then we call the union  $C \cup A$  *circles and arcs system* in  $\mathbb{R}^2$ .

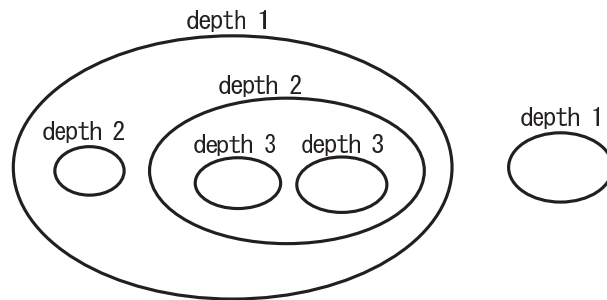


FIGURE 6. depth

Let  $S = C \cup A$  be a circles and arcs system in  $\mathbb{R}^2$ . We divide  $\mathbb{R}^2$  into regions by the circles of  $C$ . A circle  $Z$  of  $C$  is said to be of *depth 1* if  $Z$  is contained in the boundary of the infinitely large region, and of *depth  $i$*  if  $Z$  is contained in the boundary of a region  $R$  such that its boundary  $\partial R$  contains a single circle, say  $Z'$ , of depth  $i - 1$  and  $R$  is inside  $Z'$ . See Figure 6.

Every circles and arcs system can be moved to be “beautiful” as shown in the next theorem. Let  $\pi_x : \mathbb{R}^2 \ni (x, y) \mapsto (x, 0) \in \mathbb{R}^2$  and  $\pi_y : \mathbb{R}^2 \ni (x, y) \mapsto (0, y) \in \mathbb{R}^2$  be

projections. We say subsets  $A$  and  $B$  of  $\mathbb{R}^2$  *overlap each other under  $\pi_x$*  (resp.  $\pi_y$ ) if  $\pi_x(A) \cap \pi_x(B) \neq \emptyset$  (resp.  $\pi_y(A) \cap \pi_y(B) \neq \emptyset$ ).

**Theorem 1.3.** *Let  $C \cup A$  be a circles and arcs system in  $\mathbb{R}^2$ , where  $C$  is the union of circles and  $A$  the union of arcs. Then it can be deformed by an ambient isotopy of  $\mathbb{R}^2$  so that (1) circles are rectangles composed of two vertical line segments and two horizontal line segments, and (2) each arc either (a) is a vertical line segment, or (b) has its both endpoints in the same circle and is composed of three line segments: one horizontal line segment  $s$  and two vertical line segments in the same side of  $s$ . Moreover, the isotopy can be taken so that no pair of rectangular circles of the same depth overlap each other under  $\pi_y$ .*

We call a circles and arcs system *rectangular* if it satisfies the conditions (1) and (2) in the above theorem. For an example of rectangular circles and arcs system, see Figure 7. We say that an arc of  $A$  as in (2)(a) in the theorem is of *type I* and an arc as in (2)(b) is of *type  $\sqcup$* .

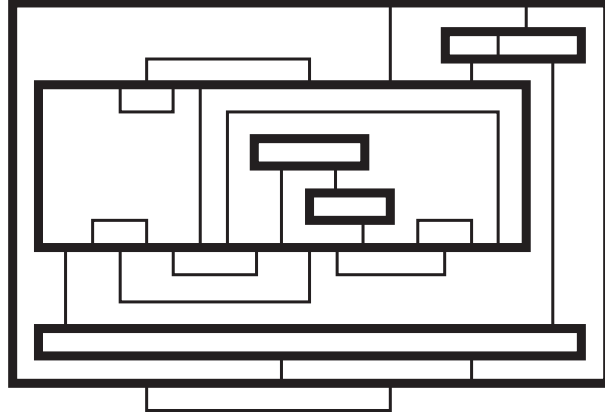


FIGURE 7. rectangular circles and arcs system

**Corollary 1.4.** *Let  $D$  be an oriented link diagram in the plane  $\mathbb{R}^2$ . The Seifert circles and arcs system for  $D$  can be deformed by an ambient isotopy of  $\mathbb{R}^2$  so that Seifert circles are rectangles composed of two vertical line segments and two horizontal line segments and arcs are vertical line segments. Moreover, the isotopy can be taken so that no pair of rectangular circles of the same depth overlap each other under  $\pi_y$ .*

For an example of deformation as in the above corollary, see Figure 8.

A circles and arcs system is called *monadic* if it has only one circle. Every connected link diagram admits a system of smoothing operations which yields a monadic circles and arcs system  $S$ . This is shown in Lemma 2.1.

Let  $S = C \cup A$  be a monadic circles and arcs system. An arc  $\beta$  with  $\beta \cap C = \partial\beta$  is called a *ruler* if it is free from the endpoints  $\partial A$ , is contained in the disk bounded by  $C$ ,

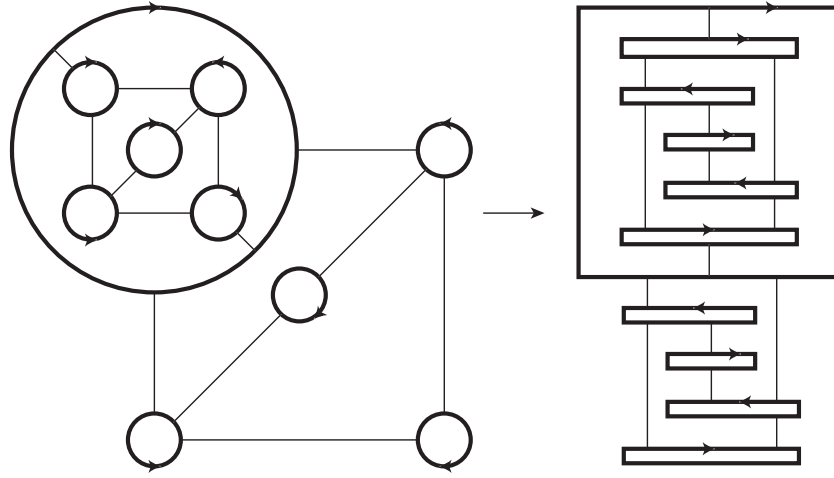


FIGURE 8. deforming a Seifert circles and arcs system to be rectangular

and intersects every arc of  $A$  transversely in at most one point. The number of intersection points of  $\beta$  and  $A$  is called the *length* of  $\beta$ . Then the *width* of  $S$  is the maximal number among lengths of all rulers for  $S$ , and let  $w(S)$  denote it. The width of a connected link diagram  $D$ , denoted by  $w(D)$ , is the maximal width  $w(S)$  over all systems of undirected smooth operations on  $D$  yielding a monadic circles and arcs system  $S$ .

The next is a corollary of Theorem 1.3 and Lemma 2.1.

**Corollary 1.5.** *Let  $D$  be a link diagram in the plane  $\mathbb{R}^2$ . Then an adequate system of undirected smoothing operations and an adequate ambient isotopy of  $\mathbb{R}^2$  deform  $D$  into a monadic circles and arcs system  $C \cup A$  such that (1) the circle  $C$  is a rectangle composed of two vertical line segments and two horizontal line segments, and (2) each arc of  $A$  is of type I or  $\sqcup$ .*

For an example of deformation as in the above corollary, see Figure 9.

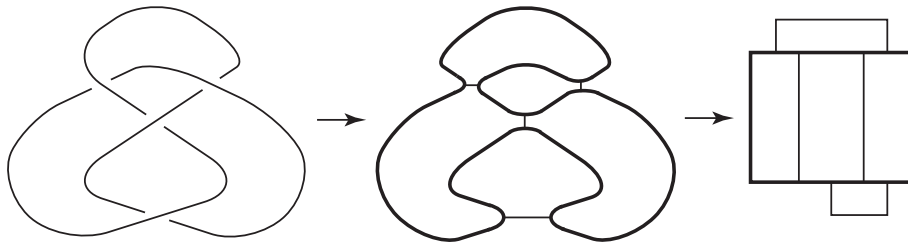


FIGURE 9. deforming a link diagram to a rectangular monadic circles and arcs system

## 2. PROOF OF THEOREM 1.2

**Lemma 2.1.** *From any connected link diagram  $D$ , we can obtain a monadic circles and arcs system by adequately applying undirected smoothing operations at all the crossings of  $D$ .*

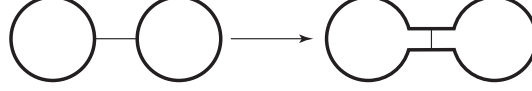


FIGURE 10. changing the way of undirected smoothing at a crossing

*Proof.* We apply arbitrary undirected smoothing operations at all the crossings of  $D$ , to obtain a circles and arcs system  $S = C \cup A$ . Note that  $S = C \cup A$  is connected because  $D$  is connected. If the set of circles  $C$  consists of a single circle, we are done. If it contains plural circles, then there is an arc of  $A$  which connects two distinct circles of  $C$ . We change the way of undirected smoothing at the crossing corresponding the arc, and obtain a circles and arcs system with one less circles. See Figure 10. Repeating this, we obtain a monadic circles and arcs system.  $\square$

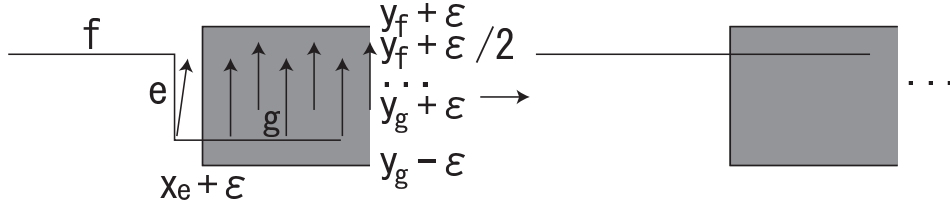


FIGURE 11. straight merge

**Lemma 2.2.** *Let  $R$  be a rectangular diagram with  $n$  vertical edges. Suppose that  $R$  has a vertical edge  $e$  such that  $e$  is free from the crossings of  $R$ , and that the horizontal edges  $f, g$  sharing an endpoint with  $e$  are in the opposite sides of  $e$  to each other. Then an adequate ambient isotopy of  $\mathbb{R}^2$  deforms  $f \cup e \cup g$  into a single horizontal edge, and  $R$  into a rectangular diagram with  $n - 1$  vertical edges.*

We call the ambient isotopy as in the proof below a *straight merge* operation at  $e$ . A similar thing holds also when  $e$  is horizontal.

*Proof.* Let  $y_f, y_g$  be the ordinates of  $f, g$  respectively, and  $x_e$  the abscissa of  $e$ . We assume, without loss of generality, that  $y_f > y_g$ , and that  $f$  is in the left side of  $e$  and  $g$  is in the right side of  $e$ . We take a small positive real number  $\epsilon$  so that there is no horizontal edge of  $R$  at any ordinate in  $(y_g - \epsilon, y_g) \cup (y_g, y_g + \epsilon) \cup (y_f, y_f + \epsilon)$  and that there is no vertical

edge of  $R$  at any abscissa in  $(x_e, x_e + \epsilon)$ . Then there is an ambient isotopy of  $\mathbb{R}^2$  as in the conclusion of this lemma which shrinks the area  $[y_g + \epsilon, y_f + \epsilon] \times [x_e + \epsilon, \infty)$  in the vertical direction so that it is deformed into the area  $[y_f + (\epsilon/2), y_f + \epsilon] \times [x_e + \epsilon, \infty)$  and expands the area  $[y_g - \epsilon, y_g + \epsilon] \times [x_e + \epsilon, \infty)$  in the vertical direction so that it is deformed into the area  $[y_g - \epsilon, y_f + (\epsilon/2)] \times [x_e + \epsilon, \infty)$ . See Figure 11.  $\square$

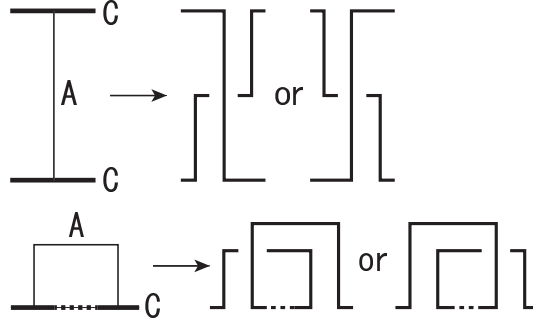


FIGURE 12. restoring the link diagram

*Proof.* We prove Theorem 1.2. Let  $D$  be a connected link diagram, and  $c(D)$  the number of crossings of  $D$ . We perform undirected smoothing operations at all the crossings so that we obtain a monadic circles and arcs system  $S = C \cup A$  with  $w(D) = w(S)$ .

Let  $\beta$  be a ruler for  $S$  such that its length gives the width  $w(D)$ . We can isotope  $S \cup \beta$  so that (1)  $\beta$  is a horizontal line segment, that (2) the circle  $C$  forms the boundary circle of a tubular neighbourhood of  $\beta$  and consists of two horizontal line segments and two vertical line segments, and that (3) arcs intersecting  $\beta$  in a single point are vertical line segments and the other arcs of  $A$  consist of two vertical line segments and a single horizontal line segment.

Then we restore the diagram  $D$  from  $S$  by replacing arcs of  $A$  with crossings as shown in Figure 12. For each of  $w(D)$  vertical arcs of  $A$ , we use a vertical line segment as an overpass, and a union of two vertical line segments and a single horizontal line segment as an underpass to form a crossing. Note that the two vertical lines are free from the crossing, and horizontal lines with which they share endpoints are in the opposite sides of them. For each of the other arcs of  $A$ , we use a union of two vertical line segments and a single horizontal line segment as both of an overpass and an underpass. Note that the three vertical lines are free from the crossing. For two of them, horizontal lines with which they share endpoints are in the opposite sides of them. Thus we obtain a link diagram composed of  $2 + 3w(D) + 4(c(D) - w(D)) = 2 + 4c(D) - w(D)$  vertical line segments.

By slightly perturbing ordinates of horizontal line segments, and abscissa of vertical line segments, we obtain a rectangular diagram with  $2 + 4c(D) - w(D)$  vertical edges. Then we perform straight merge operations at vertical edges as above, twice per each crossing as



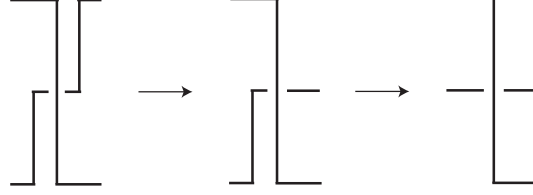


FIGURE 13. straight merge operations at vertical line segments in an underpass

shown in Figure 13, to obtain a rectangular diagram with  $(2 + 4c(D) - w(D)) - 2c(D) = 2 + 2c(D) - w(D)$  vertical edges.  $\square$

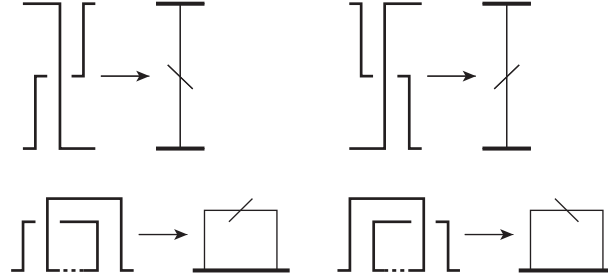


FIGURE 14. Crossings are denoted by arcs with a slash.

In the rest of this section, crossings are denoted by arcs with a slash as shown in Figure 14 when we need to specify over or under information of them.

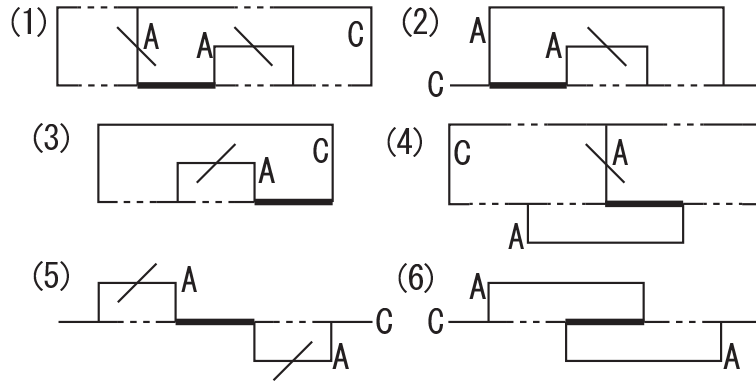


FIGURE 15. horizontal line segments at which we perform straight merge operations

After applying straight merge operations at vertical line segments twice per each crossing as in the above proof, we can perform straight merge operations at certain kinds of horizontal line segments if there are. Let  $C_h$  be the union of the top and bottom horizontal line segments of  $C$ . The endpoints of  $A$  divide  $C_h$  into shorter horizontal line segments. Among

them, those which are depicted in bold lines in Figure 15 are available for straight merge operations. For type (5), for example, the operation is performed as shown in Figure 16.



FIGURE 16. a straight merge at a horizontal line segment

In case of the circles and arcs system in Figure 17, we cannot apply straight merge operation at any horizontal line segment after the straight merge operations at vertical line segments. (However, the resulting rectangular diagram is not minimal with respect to the number of vertical edges.)

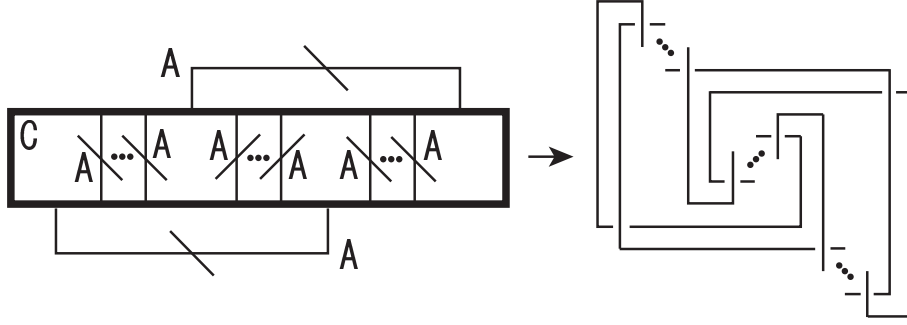


FIGURE 17. We cannot apply a straight merge at any horizontal line segment to this system.

If there is an arc of type  $\sqcup$  inside  $C$ , and the link diagram does not have a monogon face, then we can apply a straight merge at a horizontal line segment as in the lemma below. Let  $\alpha$  be an arc of type  $\sqcup$  inside  $C$  (resp. outside  $C$ ), and  $Q$  the disk bounded by the circle formed by  $\alpha$  and a subarc of  $C_h$ . We call  $\alpha$  is *innermost* among the arcs of type  $\sqcup$  inside  $C$  (resp. outside  $C$ ) if  $Q$  does not contain such an arc other than  $\alpha$ .

**Lemma 2.3.** Let  $D, S, C, A$  as in the proof of Theorem 1.2,  $C_h$  as above, and  $m$  the number of innermost arcs of type  $\sqcup$  inside  $C$ . Suppose that  $D$  has no monogon region. After performing straight merge operations at vertical line segments twice per each arc of  $A$ , we can apply at least  $m$  straight merge operations at horizontal line segments as in Figure 15 (6).

*Proof.* Let  $\alpha$  be an arc of type  $\sqcup$  which is innermost inside  $C$ , and  $\beta$  the subarc of  $C_h$  between the endpoints  $\partial\alpha$ . If  $\text{int}\beta$  is disjoint from endpoints of  $A$ , then  $D$  has a monogon region, a contradiction. Hence  $\text{int}\beta$  contains an endpoint of  $A$ . Let  $\gamma$  be an innermost arc

of  $A$  outside  $C$  with at least one of its endpoints in  $\beta$ . If  $\gamma$  has both its endpoints in  $\beta$ , then  $\gamma$  gives a monogon region of  $D$ , which is a contradiction. Hence  $\gamma$  has precisely one of its endpoints in  $\beta$ . Then  $\alpha$  and  $\gamma$  together form the pattern (6) in Figure 15, and we can apply a straight merge operation there.  $\square$

### 3. PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1 using Corollary 1.4. (Corollary 1.4 is a corollary of Theorem 1.3 which is proven in the next section.)

*Proof.* We prove Theorem 1.1. Let  $D$  be an oriented link diagram, and  $c(D)$  and  $s(D)$  the numbers of crossings and Seifert circles of  $D$  respectively. We perform smoothing operations at all the crossings, so that we obtain a Seifert circles and arcs system  $S = C \cup A$ . Using Corollary 1.4, we can deform  $S$  by an ambient isotopy of  $\mathbb{R}^2$  into a rectangular Seifert circles and arcs system  $R$ . We restore the diagram  $D$  from  $R$  by replacing arcs of  $A$  with crossings as shown in the upper half of Figure 12. The resulting link diagram has  $2s(D) + 3c(D)$  vertical line segments. Then we perform straight merge operations as described in the proof of Lemma 2.2 twice per each crossing as shown in Figure 13. Note that each straight merge operation decreases the number of vertical line segments by one. After an adequate small ambient isotopy of  $\mathbb{R}^2$ , no pair of line segments are colinear. Thus we obtain a rectangular diagram with  $2s(D) + 3c(D) - 2c(D) = 2s(D) + c(D)$  vertical edges.

We consider the case where  $c(D) \geq 1$  and  $D$  has no nugatory crossings. If  $c(D) = 1$ , then  $D$  would have a nugatory crossing. Hence  $c(D) \geq 2$ . We observe the leftmost arc  $\alpha_l$  of  $A$ . In the argument in the previous paragraph, we have performed straight merge operations twice at two vertical line segments in underpass of the crossing corresponding to  $\alpha_l$ . The second arrows in Figure 18 show the deformation as above. Now, we perform deformations described by the third and the forth arrows in Figure 18. Note that  $\alpha_l$  has its endpoints in distinct circles of  $C$  since  $R$  is a Seifert circles and arcs system. In Cases (1) and (2) in Figure 18,  $\alpha_l$  has its endpoints in circles of the same depth, while it does not in Cases (3) and (4). In Cases (1), (2) and (4) in Figure 18, we apply a straight merge operation at a horizontal line segment drawn in a bold line. In Case (3) in Figure 18, we shrink circles of  $C$  which overlap  $\alpha_l$  under  $\pi_x$  and do not contain an endpoint of  $\alpha_l$  to the right direction so that they are in the right side of the vertical line segment forming the overpass of the crossing corresponding  $\alpha_l$ , and then, we perform a “generalized merge operation” as shown by the forth arrow. We do a similar operation for the rightmost arc of  $A$ , to obtain a rectangular diagram with  $2s(L) + c(L) - 2$  vertical edges.  $\square$

### 4. RECTANGULAR CIRCLES AND ARCS SYSTEM

We prove Theorem 1.3 in this section.

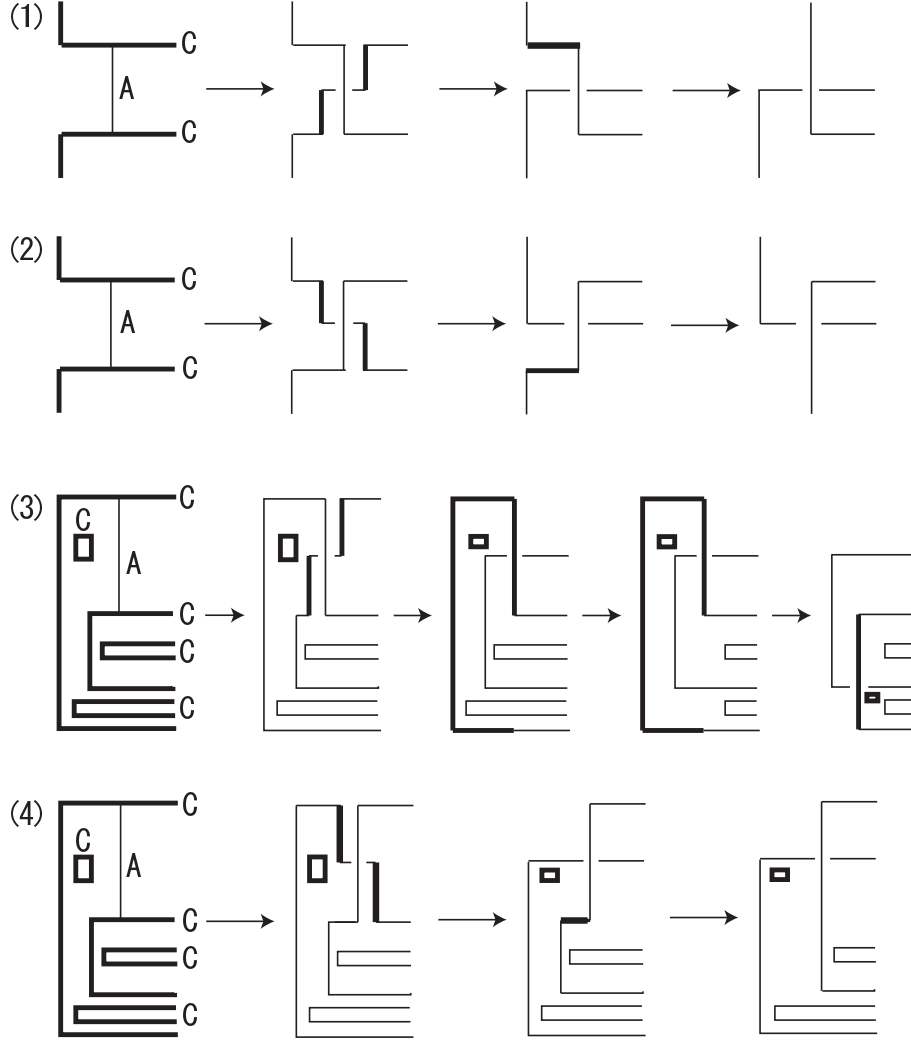


FIGURE 18. straight merge or generalized merge

*Definition 4.1.* Let  $S = C \cup A$  be a rectangular circles and arcs system with  $C$  being the union of circles and  $A$  the union of arcs. Let  $R$  be a rectangular disk whose boundary circle  $\partial R$  is composed of two vertical line segments and two horizontal line segments. We call  $S \cap R$  a *black box* in  $S$  if  $\partial R \cap C = \emptyset$ , the vertical lines of  $\partial R$  are disjoint from  $S$ , the horizontal lines of  $\partial R$  intersect  $A$  transversely in one or more points.

**Lemma 4.2.** *Let  $S, C, A, R$  be as in Definition 4.1 above. Let  $\tilde{R}$  be a rectangular disk such that it forms a regular neighbourhood of  $R$ , that its boundary circle  $\partial \tilde{R}$  is composed of two vertical line segments and two horizontal line segments, and that  $S \cap (\tilde{R} - \text{int } R)$  consists of vertical line segments, say,  $\gamma_1, \gamma_2, \dots, \gamma_k$  from the right above the top horizontal line segment of  $\partial R$ , and  $\alpha_1, \alpha_2, \dots, \alpha_j, \beta_1, \beta_2, \dots, \beta_i$  from the right below the bottom horizontal line segment of  $\partial R$  for some non-negative integers  $i, k$  and a positive integer  $j$ .*

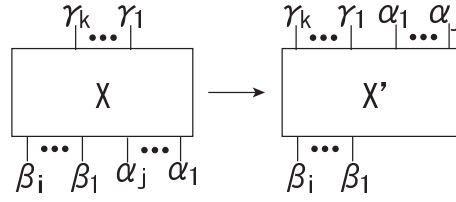


FIGURE 19. Deformation in Lemma 4.2

Suppose that there is no vertical line segment in  $A$  connecting one of  $\gamma_1, \dots, \gamma_k$  and one of  $\alpha_1, \dots, \alpha_j, \beta_1, \dots, \beta_i$ , and that no pair among  $\alpha_1, \dots, \alpha_j, \beta_1, \dots, \beta_i$  is contained in the same arc of type  $\sqcup$ . See Figure 19. Then there is an ambient isotopy  $H : \mathbb{R}^2 \times [0, 1] \rightarrow \mathbb{R}^2$  with a homeomorphism  $H_t : \mathbb{R}^2 \ni x \mapsto H_t(x) = H((x, t)) \in \mathbb{R}^2$  for all  $t \in [0, 1]$  such that  $H_t(R) = R$  and  $H_t(\tilde{R}) = \tilde{R}$  for all  $t \in [0, 1]$ , that  $H_t(p) = p$  for all points  $p$  in the left vertical line segments of  $\partial R \cup \partial \tilde{R}$  and for all  $t \in [0, 1]$ , that  $H_1(\alpha_m)$  is a vertical line segment above the top horizontal line segment of  $\partial R$  for all  $m \in \{1, 2, \dots, j\}$ , and that  $H_1(S \cap R)$  is a black box in some rectangular circles and arcs system.

*Remark 4.3.* This lemma does not mention deformation of  $S$  outside  $\tilde{R}$  which should occur accompanied by the deformation within  $\tilde{R}$ . The isotopy in this lemma may not keep  $S$  rectangular.

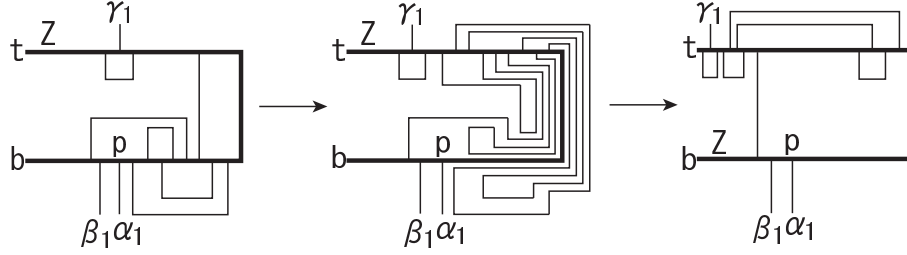
*Proof.* We say that an ambient isotopy  $H : \mathbb{R}^2 \times [0, 1] \rightarrow \mathbb{R}^2$  is *good* if  $H_t(R) = R$  and  $H_t(\tilde{R}) = \tilde{R}$  for all  $t \in [0, 1]$ , and  $H_t(p) = p$  for all points  $p$  in the vertical line segments of  $\partial R \cup \partial \tilde{R}$  and for all  $t \in [0, 1]$ .

We can move the circles and arcs system  $S$  by a good ambient isotopy of  $\mathbb{R}^2$  so that the rectangular circles of  $C$  are thinned in the vertical direction, and no pair of rectangular circles of the same depth overlap under  $\pi_y$ , since the arcs of  $A$  intersects  $C$  in endpoints of vertical line segments in  $A$ .

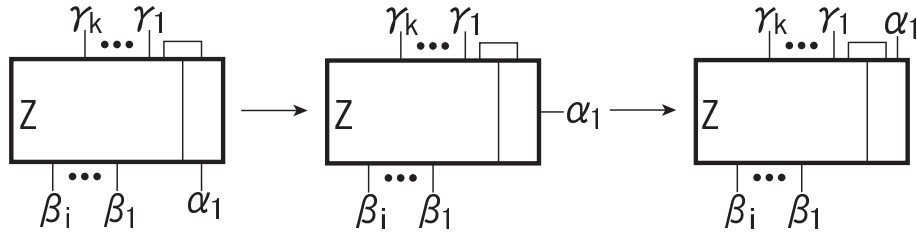
It is enough to show this lemma in the case  $j = 1$ . Applying the result for the case  $j = 1$  repeatedly, we obtain the desired conclusion also for the case  $j > 1$ .

Set  $X = S \cap R$ , the black box. The vertical line segment of  $A$  containing  $\alpha_1$  has an endpoint  $p$  in the bottom horizontal line segment  $b$  of some circle  $Z$  of  $C$  in  $X$ . The point  $p$  divides  $b$  into two segments, one of which, say  $b_r$ , lies in the right side of  $p$ . Let  $t$  be the top horizontal line segment of  $Z$ .

Let  $n_C(X)$  denote the number of circles of  $C \cap X$ . The proof proceeds by induction on  $n_C(X)$ . We first consider the case  $n_C(X) = 1$ . Then  $Z$  is the only circle of  $C$  contained in  $X$ . We move arcs of  $A$  by a good ambient isotopy of  $\mathbb{R}^2$ . If the interior of  $b_r$  contains endpoints of arcs of  $A$ , we move the arcs near the endpoints along a subarc of  $b_r$  and the right vertical line segment of  $Z$  so that the endpoints are contained in the top horizontal line segment of  $Z$ . See Figure 20. Because  $j = 1 > 0$ , no arc of  $A$  outside  $Z$  has an endpoint

FIGURE 20. The case  $c(X) = 1$ 

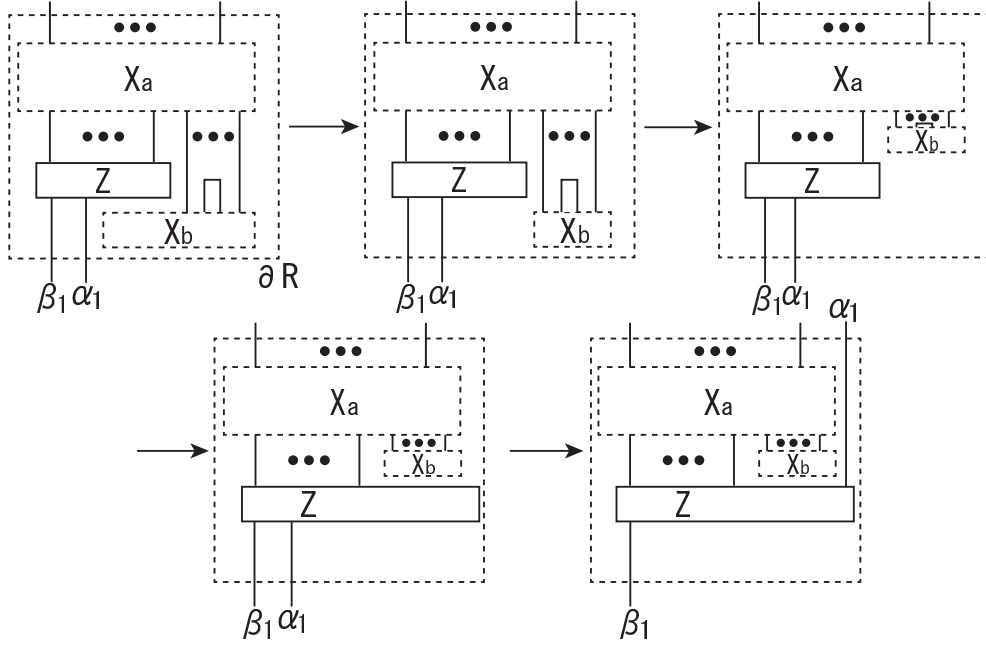
in  $t$  and the other in  $b - b_r$  or below the bottom horizontal line segment of  $\partial R$ . Any arc with both endpoints in  $t$  cobounds a disk with a subarc of  $t$  inside or outside  $Z$ . Then we move arcs of  $A$  so that  $A \cap R$  is a union of arcs of types I and  $\sqcup$ . There is no obstruction because  $n_C(X) = 1$ . Thus we can assume that  $\text{int } b_r$  does not contain such an endpoint of  $A$ . Then we can move  $\alpha_1$  so that it is above the top horizontal line segment of  $Z$  as shown in Figure 21, which shows the lemma in the case of  $n_C(X) = 1$ .

FIGURE 21. Bring  $\alpha_1$  above  $t$ 

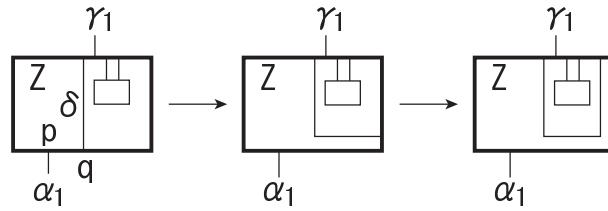
We consider the case  $n_C(X) > 1$ . Let  $y_t, y_b$  (resp.  $\eta_t, \eta_b$ ) be the ordinates of the top and the bottom horizontal line segments of  $Z$  (resp.  $\partial R$ ), and  $x_l, x_r$  (resp.  $\xi_l, \xi_r$ ) the abscissae of the left and the right vertical line segments of  $Z$  (resp.  $\partial R$ ). Let  $x_\alpha$  be the abscissa of  $\alpha_1$ . We consider rectangles

$R_a = [\xi_l + \epsilon, \xi_r - \epsilon] \times [y_t + \epsilon, \eta_t - \epsilon]$  and  $R_b = [x_\alpha + \epsilon, \xi_r - \epsilon] \times [\eta_b + \epsilon, y_b - \epsilon]$ , and black boxes  $X_a = S \cap R_a$  and  $X_b = S \cap R_b$ , where  $\epsilon$  is a small positive real number such that there is no horizontal line segment in  $S$  with ordinate in the union of open intervals  $(\eta_b, \eta_b + \epsilon) \cup (y_b - \epsilon, y_b) \cup (y_t, y_t + \epsilon) \cup (\eta_t - \epsilon, \eta_t)$  and there is no vertical line segment in  $S$  with abscissa in  $(\xi_l, \xi_l + \epsilon) \cup (x_\alpha, x_\alpha + \epsilon) \cup (\xi_r - \epsilon, \xi_r)$ .

We consider first the case where  $\text{int } b_r$  does not contain an endpoint of an arc of  $A$  outside  $Z$ . In this case, the proof proceeds by induction on the number, say  $n_e$ , of endpoints of arcs of  $A$  contained in  $\text{int } b_r$ . Such arcs are inside  $Z$ . When  $n_e = 0$ , we deform  $S$  as in Figure 22. We shrink  $X_b$  in the horizontal direction by a good ambient isotopy of  $\mathbb{R}^2$  so that it is contained in  $(x_r, \xi_r - \epsilon] \times [\eta_b + \epsilon, y_b - \epsilon]$ . Then, shrinking the arcs connecting  $X_b$  and

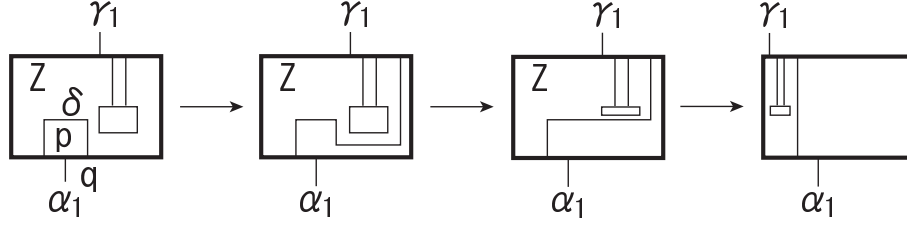
FIGURE 22. The case where  $\text{int } b_r$  is free from endpoints

$X_a$  and thinning  $X_b$  in the vertical direction, we move  $X_b$  upward so that the ordinates of points in  $X_b$  are within the interval  $(y_t, y_t + \epsilon)$ . Now, there is nothing in the right side of  $Z$  within  $R$ . We lengthen  $Z$  to the right direction so that the abscissa of the right vertical line segment of  $Z$  is a little larger than  $\xi_r - \epsilon$ . Then we can move the arc  $\alpha_1$  so that it forms a vertical line segment above the top horizontal line segment of  $Z$ .

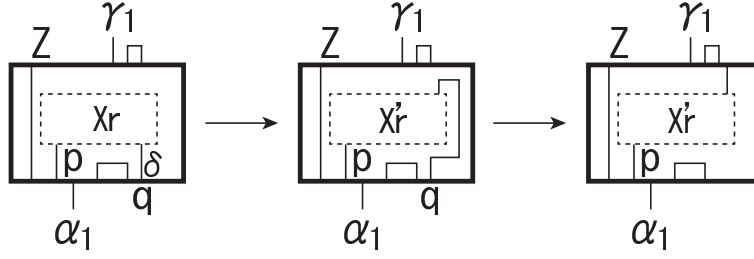
FIGURE 23. The case where  $\delta$  is of type I and with  $\partial\delta$  in  $Z$ 

We consider the case  $n_e > 0$ . Let  $q$  be the rightmost endpoint among those of arcs of  $A$  in  $\text{int } b_r$ , and  $\delta$  the arc with  $q \in \partial\delta$ . Note that  $\delta$  is inside  $Z$ . If  $\delta$  is an arc of type I with its both endpoints in  $Z$ , then we move  $\delta$  near the point  $q$  along a subarc of  $b_r$  and the right vertical line segment of  $Z$ , so that  $\delta$  is deformed into an arc of type  $\sqcup$ . See Figure 23.

If  $\delta$  is an arc of type  $\sqcup$ , then we can move  $\delta$  as shown in Figure 24 so that it forms an arc of type I. We deform  $\delta$  near the point  $q$  along a subarc of  $b_r$  and the right vertical line

FIGURE 24. The case where  $\delta$  is of type  $\sqcup$ 

segment of  $Z$ , and then perform deformations similar to straight merges as described in the proof of Lemma 2.2.

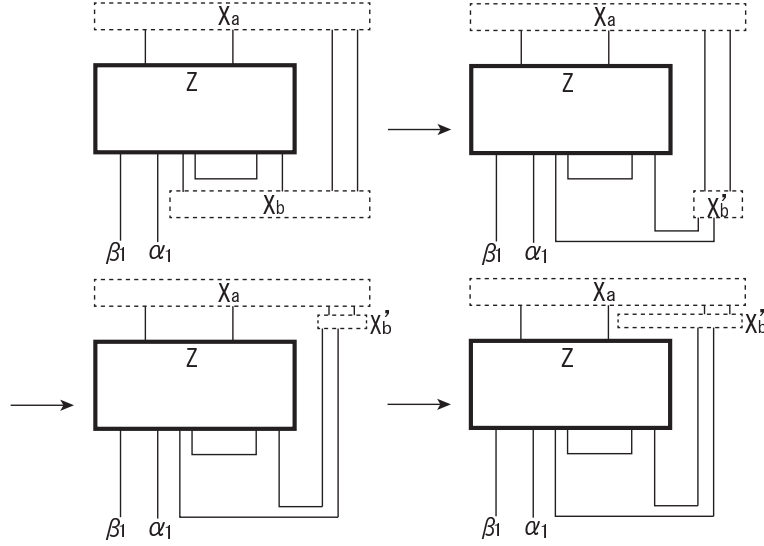
FIGURE 25. The case where  $\delta$  connects  $Z$  and other circle of  $C$ 

In the remaining case,  $\delta$  has the other endpoint in a circle of  $C$  other than  $Z$ . Let  $R_Z$  be the rectangular disk bounded by  $Z$  in  $\mathbb{R}^2$ . The arcs of type I with their both endpoints in  $Z$  divides  $R_Z$  into subdisks. Let  $R'_r$  be the rightmost one with  $R'_r = [x_I, x_r] \times [y_b, y_t]$  for some real number  $x_I$ . If there are no such arc of type I, then we set  $x_I = x_l$  and  $R'_r = R_Z$ . Then, set  $R_r = [x_I + \epsilon', x_r - \epsilon'] \times [y_b + \epsilon', y_t - \epsilon']$ , where the positive real number  $\epsilon'$  is taken to be small so that there is no horizontal line segment in  $S$  with ordinate in  $(y_b, y_b + \epsilon') \cup (y_t - \epsilon', y_t)$  and there is no vertical line segment in  $S$  with abscissa in  $(x_I, x_I + \epsilon') \cup (x_r - \epsilon', x_r)$ . Let  $X_r = S \cap R_r$ , the black box. If  $X_r$  contains an arc  $\lambda$  of type  $\sqcup$  with its both endpoints in  $b$ , then we deform the rectangle cobounded by  $\lambda$  and a subarc of  $b$  to be very thin in the vertical direction so that  $\lambda$  gets out from  $X_r$ . See Figure 25. Note that  $n_C(X_r) < n_C(X)$ . By the hypothesis of induction, we can apply Lemma 4.2 to  $X_r$ . Then the black box  $X_r$  is deformed to some black box, say  $X'_r$ , and the arc  $\delta$  is deformed to an arc which connects  $b_r$  and the top horizontal line segment of  $\partial R_r$ . Then  $\delta$  can be deformed to a vertical line segment connecting the top horizontal line segment of  $Z$  and that of  $\partial R_r$ .

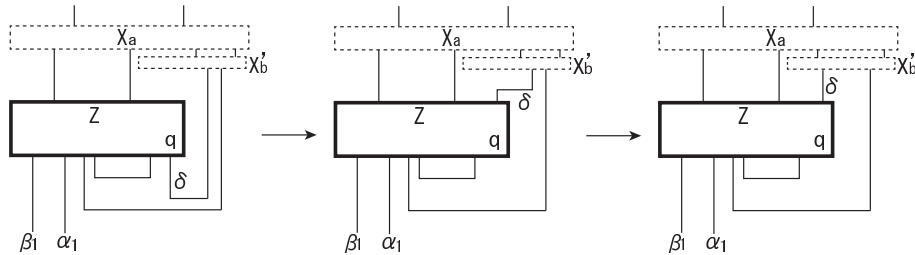
In each of the three cases above we can decrease  $n_e$ , and the lemma follows by induction.

We consider the case where the interior of  $b_r$  contains an endpoint of an arc of  $A$  outside  $Z$ . If  $X_b$  contains an arc  $\mu$  of type  $\sqcup$  with its both endpoints in  $\text{int } b_r$ , then we deform the rectangle cobounded by  $\mu$  and a subarc of  $b_r$  to be very thin in the vertical direction so that  $\mu$  gets out from  $X_b$ . Note that  $n_C(X_b) < n_C(X)$ . The hypothesis of induction allows



FIGURE 26. The case where  $\delta$  is outside  $Z$ 

us to apply Lemma 4.2 to  $X_b$  so that the vertical line segments connecting  $b$  and the top line segment of  $\partial R_b$  are deformed to be arcs connecting  $b$  and the bottom line segment of  $\partial R_b$ . Let  $X'_b$  be the black box obtained from  $X_b$  by this deformation. See Figure 26. We move  $X'_b$  as in this figure. We shrink  $X'_b$  so that the abscissa of one of the leftmost points of  $X'_b$  is larger than that of the right vertical line segment of  $Z$ , and lift  $X'_b$  up so that the ordinate of one of the bottom points of  $X'_b$  is larger than that of the top horizontal line segment of  $Z$ , and then lengthen it so that the abscissa of one of the leftmost points of  $X'_b$  is smaller than that of the right vertical line segment of  $Z$ . At this stage, the circles and arcs system is not rectangular since the arcs connecting  $b$  and the bottom line segment of the boundary of the rectangle bounding  $X'_b$  are neither of type I nor of type  $\sqcup$ . We will deform them into arcs of type I one by one from now.

FIGURE 27. The case where  $\delta$  is outside  $Z$  and of type I

Let  $q$  be the rightmost endpoint among those of arcs of  $A$  in  $b_r$ . If  $q$  is an endpoint of an arc inside  $Z$ , then we perform one of deformations as in Figures 23 through 25 above. We

consider the case where  $q$  is an endpoint of an arc  $\delta$  outside  $Z$ . If  $\delta$  connects  $Z$  and another circle of  $C$ , then we can deform  $\delta$  to be a vertical line segment connecting  $Z$  and  $X'_b$  as in Figure 27. The last deformation in this figure is similar to the straight merge described in the proof of Lemma 2.2.

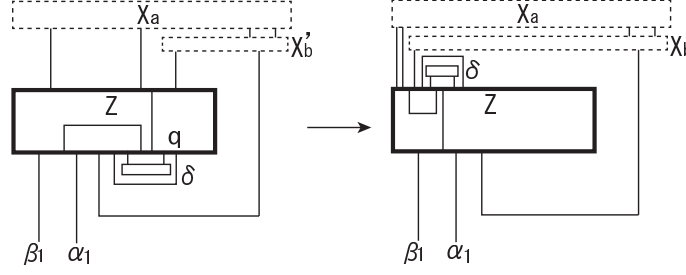


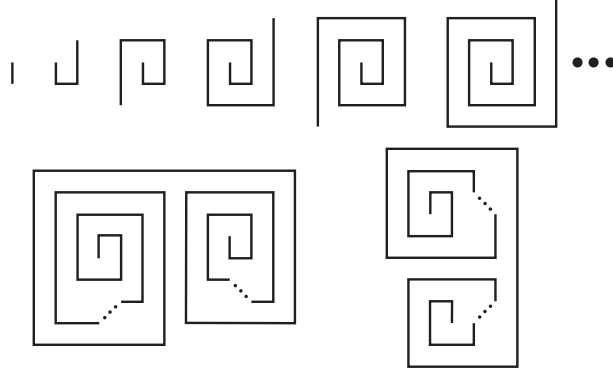
FIGURE 28. The case where  $\delta$  is outside  $Z$  and of type  $\sqcup$

When  $\delta$  has its both endpoints in  $Z$  and is of type  $\sqcup$ , a subarc of  $b_r$  and  $\delta$  cobound a disk, say  $R_{\sqcup}$ . We shrink  $R_{\sqcup}$  to be very small, and move it along a subarc of  $b_r$  and the right vertical line segment of  $Z$  as in Figure 28. If the arc  $R_{\sqcup} \cap b_r$  contains endpoints of arcs of  $A$  inside  $Z$ , then we also deform inside  $Z$  as in Figures 23 through 25.

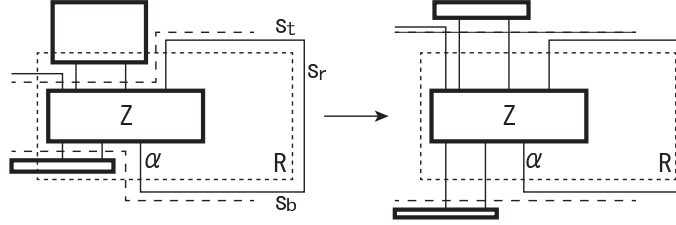
Repeating such deformations,  $\text{int } b_r$  becomes free from endpoints of  $A$  and  $S$  becomes rectangular again. Then, similarly to Figure 22, we lengthen  $Z$  to the right direction, and move  $\alpha_1$  so that it forms a vertical line segment above the top horizontal line segment of  $Z$ . This completes the proof of the lemma.  $\square$

*Proof.* We prove Theorem 1.3. The proof proceeds by induction on the number of arcs of  $A$ . If  $A$  is empty, then the theorem is very clear. We assume that the theorem holds when  $A$  consists of  $n - 1$  arcs, and consider the case where  $A$  consists of  $n$  arcs. Let  $A'$  be the union of arbitrary  $n - 1$  arcs of  $A$ . We can deform the circles and arcs system  $S = C \cup A$  by an ambient isotopy of  $\mathbb{R}^2$  so that  $C \cup A'$  is rectangular, and that the arc  $\alpha = A - A'$  is composed of  $m$  vertical line segments and  $m - 1$  horizontal line segments for some positive integer  $m$ . Note that  $A \cap C (= \partial A)$  are endpoints of vertical line segments in  $A$ . Hence we can deform the rectangular circles of  $C$  very thin in the vertical direction so that no pair of rectangular circles of the same depth overlap under  $\pi_y$ .

If  $\alpha$  has a vertical line segment (resp. horizontal line segment)  $e$ , which has two horizontal line segments (resp. vertical line segments) sharing an endpoint with  $e$  in both sides of  $e$ , then we can perform a deformation similar to the straight merge operation in the proof of Lemma 2.2, to decrease the number of line segments forming  $\alpha$ . Hence, without loss of generality, we assume that  $\alpha$  does not contain such a line segment and is of one of the forms shown in Figure 29 or images of them by a reflection in a vertical line or a horizontal line, a rotation through  $180^\circ$  or their composition. Thus, if the arc  $\alpha$  has its both endpoints in the

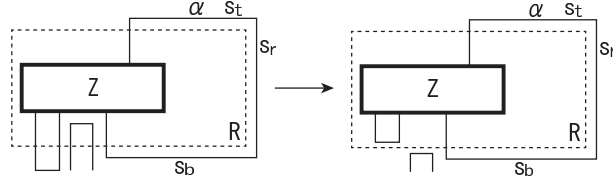
FIGURE 29.  $\alpha$ 

same circle of  $C$  and is inside the circle, then  $\alpha$  is already of type I or  $\sqcup$ , and the theorem follows. Hence we can assume that either (1)  $\alpha$  has its both endpoints in the same circle, say  $Z$ , and is outside  $Z$ , or (2)  $\alpha$  connects distinct circles of  $C$ .

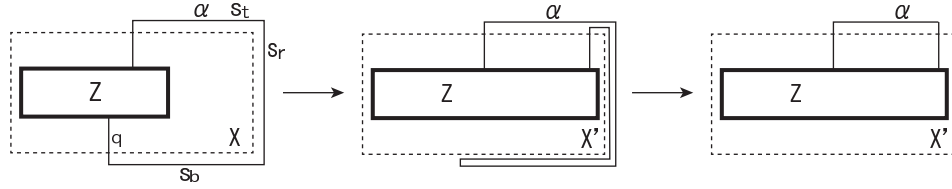
FIGURE 30. get the circles intersecting  $\partial R$  out of  $R$ 

We consider first Case (1). Then  $m = 2, 3$  or  $4$ . If  $m = 2$ , then  $\alpha$  is of type  $\sqcup$ , and we are done. When  $m = 3$ , the arc  $\alpha$  has two horizontal line segments, say  $s_t$  and  $s_b$ , with their ordinates  $t_\alpha, b_\alpha$  satisfying  $t_\alpha > b_\alpha$ . See Figure 30. We can assume, without loss of generality, that the vertical line segment, say  $s_r$ , in  $\alpha$  between  $s_t$  and  $s_b$  connects right endpoints of  $s_t$  and  $s_b$ . Let  $\ell_Z$  be the abscissa of the left vertical line segment of  $Z$ ,  $r_\alpha$  the abscissa of the vertical line segment  $s_r$ , and  $R$  the rectangular disk  $[\ell_Z - \epsilon, r_\alpha - \epsilon] \times [b_\alpha + \epsilon, t_\alpha - \epsilon]$  for a very small positive real number  $\epsilon$ . We take  $\epsilon$  so that there is no vertical line segment with its abscissa in  $(\ell_Z - \epsilon, \ell_Z) \cup (r_\alpha - \epsilon, r_\alpha)$ , and there is no horizontal line segment with its ordinate in  $(b_\alpha, b_\alpha + \epsilon) \cup (t_\alpha - \epsilon, t_\alpha)$ .

If  $\partial R$  intersects circles of  $C$  other than  $Z$  or the left vertical line segment of  $\partial R$  intersects horizontal line segments in  $A$ , then we perform deformations at left parallel copies of the two vertical line segments  $\alpha - (s_b \cup s_r \cup s_t) - \text{int } N(\partial\alpha)$  similar to the straight merge in the proof of Lemma 2.2 to get such circles and horizontal line segments out of  $R$ , where  $N(\partial\alpha)$  is a very small regular neighborhood of  $\partial\alpha$  in  $\alpha$ . See Figure 30. If the bottom horizontal line segment of  $\partial R$  intersects arcs of type  $\sqcup$ , then we shrink them and what are surrounded

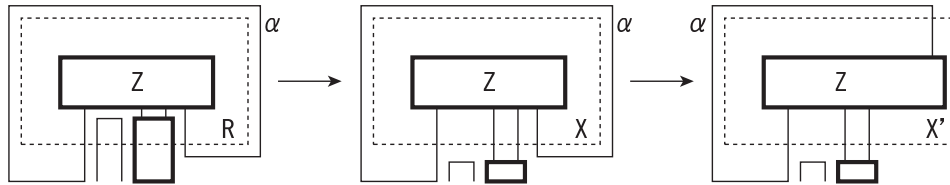
FIGURE 31. get the arcs of type  $\sqcup$  away from  $\partial R$ 

by them and subarcs of  $C$  in the vertical direction (upward or downward) to cancel the intersection points. See Figure 31.

FIGURE 32. Case (1),  $m = 3$ 

Let  $q$  be the intersection point of  $\alpha$  and the bottom horizontal line segment of  $\partial R$ . We can apply Lemma 4.2 to the black box  $X = S \cap R$ , to bring  $q$  to the top horizontal line segment of  $\partial R$ . See Figure 32. Then we can deform  $\alpha$  to an arc of type  $\sqcup$ , and the theorem follows in this case.

When  $m = 4$ , a similar argument as above decreases  $m$  to 3, and then the theorem follows by the above argument. See Figure 33.

FIGURE 33. Case (1),  $m = 4$ 

We consider Case (2). We proceed by induction on the number  $m$  of vertical line segments in  $\alpha$ . If  $m = 1$ , then  $S$  is rectangular, and we are done. Hence we assume that  $m \geq 2$ , and that the theorem holds if  $\alpha$  has at most  $m - 1$  vertical line segments as the hypothesis of induction.

If the two circles of  $C$  connected by  $\alpha$  are of distinct depths, then let  $Z$  be the circle of larger depth. If  $\alpha$  connects two circles of  $C$  of the same depth, then we take  $Z$  as below. When  $m \geq 3$ , let  $Z$  be the circle of  $C$  which contains one of the endpoints of  $\alpha$  whose

abscissa is between abscissae of some two vertical line segments in  $\alpha$ . We can assume, without loss of generality, that  $\alpha$  has an endpoint in the bottom horizontal line segment of  $Z$  rather than in the top horizontal line segment. When  $m = 2$ , we can assume, without loss of generality, both endpoints of  $\alpha$  are contained in the bottom horizontal line segment of circles of  $C$ . In this case, let  $Z$  be one of the circles connected by  $\alpha$  such that the ordinate of the top horizontal line segment of  $Z$  is smaller than that of the bottom horizontal line segment of the other circle. (Recall that circles of the same depth do not overlap under  $\pi_y$ .)

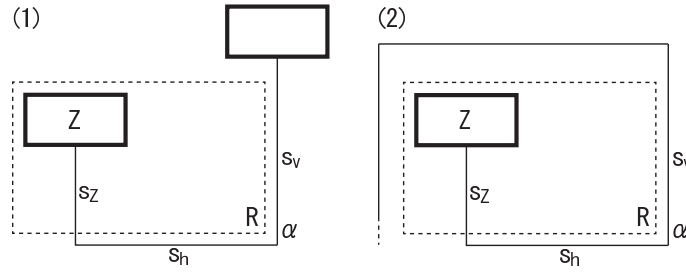


FIGURE 34. Case (2)

Let  $s_Z$  be the vertical line segment in  $\alpha$  with one of its endpoints in  $Z$ ,  $s_h$  the horizontal line segment in  $\alpha$  which share an endpoint with  $s_Z$ , and  $s_v$  the vertical line segment in  $\alpha$  other than  $s_Z$  such that  $s_v$  and  $s_h$  share an endpoint. We assume, without loss of generality, that  $s_Z \cap s_h$  is the left endpoint of  $s_h$ . Let  $x_1$  be the abscissa of the left vertical line segment of  $Z$ ,  $x_2$  the abscissa of the vertical line segment  $s_v$ ,  $y_1$  the ordinate of the horizontal line segment  $s_h$ ,  $y_2$  the ordinate of the top horizontal line segment of  $Z$ , and  $R$  the rectangular disk  $[x_1 - \epsilon, x_2 - \epsilon] \times [y_1 + \epsilon, y_2 + \epsilon]$  in  $\mathbb{R}^2$  for a very small real number  $\epsilon$ . We take  $\epsilon$  so that  $S$  has no vertical line segment with its abscissa in  $(x_1 - \epsilon, x_1) \cup (x_2 - \epsilon, x_2)$ , and  $S$  has no horizontal line segment with its ordinate in  $(y_1, y_1 + \epsilon) \cup (y_2, y_2 + \epsilon)$ . The top horizontal line segment and the right vertical line segment of  $\partial R$  do not intersect a circle of  $C$ . See Figure 34. If  $\partial R$  intersects circles of  $C$  or the left vertical line segment of  $\partial R$  intersects horizontal line segments in  $A$ , then we perform a deformation at left parallel copy of  $s_Z - \text{int } N(\partial\alpha)$  similar to the straight merge as in the proof of Lemma 2.2, to get such circles of  $C$  and horizontal line segments out of  $R$ . If the bottom horizontal line segment of  $\partial R$  intersects arcs of type  $\sqcup$ , then we shrink them and what are surrounded by them and subarcs of  $C$  in the vertical direction (upward or downward) to cancel the intersection points. Then we can deform  $S$  as shown in Figure 35. Precisely, let  $q$  be the intersection point  $\alpha \cap \partial R$ . We can apply Lemma 4.2 to the black box  $X = S \cap R$ , to bring  $q$  to the top horizontal line segment of  $\partial R$ . Note that  $Z$  is lengthened to the right by the isotopy in the proof of Lemma 4.2. Then we can deform  $\alpha$  to an arc with less number of vertical line segments. The theorem follows by the hypothesis of induction.  $\square$

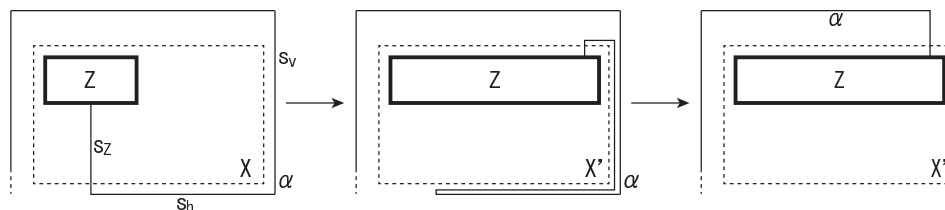


FIGURE 35. deformation in Case (2)

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## REFERENCES

- [1] J.S. Birman and W.W. Menasco, *Special positions for essential tori in link complements*, Topology **33** (1994) 525–556.
- [2] P.R. Cromwell, *Embedding knots and links in an open book I: Basic properties*, Topology and its Applications **64** (1995), 37–58.
- [3] I.A. Dynnikov, *Arc-presentations of links: Monotone simplification*, Fundamenta Mathematicae **190** (2006), 29–76.
- [4] I.A. Dynnikov, *Recognition algorithms in knot theory*, Uspekhi Mat. Nauk **58** (2003), 45–92; translation in Russian Math. Surveys **58** (2003), 1093–1139.
- [5] A. Henrich and L. Kauffman, *Unknotting Unknots*, preprint (arXiv:1006.4176v4 [math.GT] 4Nov2011).
- [6] H. Seifert, *Über das Geschlecht von Knoten*, Math. Ann. **110** (1934), 571–592.